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Irrational Involutions on Algebraic Curves.

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§ 1. *General Definitions.*

Given a correspondence between the points of two algebraic plane curves, $F(x_1, x_2, x_3) \equiv F(x) = 0$, of genus p , and $f(y) = 0$, of genus π , such that to an arbitrary point P of the first curve correspond b points of the second, and to an arbitrary point P' of the second correspond a points of the first. The group of b points on $f(y) = 0$, corresponding to P in its turn fixes on $F(x) = 0$ b groups of a points each, and P belongs to each of these groups. Similarly, P' on $f(y) = 0$ belongs to a groups, each consisting of b points. The numbers a, b indicating the number of groups in the correspondence to which arbitrary points on $f(y) = 0$ and $F(x) = 0$ belong, are called the *indices* of the (a, b) correspondence, so that the correspondence on $F(x) = 0$ is of index b , on $f(y) = 0$ of index a .

Every (a, b) correspondence can be expressed by means of two equations between the coordinates of corresponding points, if the two curves are irrational. There is usually a finite number of pairs of points not belonging to the correspondence, the coordinates of which satisfy the equations. If the curves are rational the correspondence can always be expressed by means of one equation. If a point x on $F(x) = 0$ is defined in terms of a parameter λ , and a point y on $f(y) = 0$ in terms of μ , then the (a, b) correspondence may be defined by the polynomial $\Sigma \phi_i(\lambda) \cdot \mu^{b-i}$, in which each $\phi_i(\lambda)$ is a polynomial of order a in λ . It will be stated later when this can be done even when the curves are irrational.

If either a or b is one, the coordinates of P or P' can be expressed rationally in terms of the coordinates of P' or P . If both a and b are equal to one, that is if the correspondence is one-to-one, then it is birational, and the coordinates of points on each curve can be expressed rationally in terms of the coordinates of points on the other. In this case the curves $F(x) = 0, f(y) = 0$ are said to be birationally equivalent.

A correspondence of index 1, for instance $(a, 1)$, is called an *involution* of order a . The genus of the curve, the points of which are in $(1, a)$ corre-

spondence with the points of the given curve, is called the *genus of the involution*. If the genus of the involution is zero it is said to be rational; otherwise, irrational.

Every rational involution is a linear series and vice versa.

If a curve possesses two rational or irrational involutions, $\gamma'_m, \mu'_n (m \leq n)$ such that every point of a group of the first belongs to a distinct group of the second, in which case the groups of μ'_n , taken m at a time are conjugate under γ'_m , we say that γ'_m is compounded with μ'_n .

Given a curve possessing a γ'_t . If an arbitrary point determines not 1, but k distinct groups of the involution, the involution is said to be multiple.

§ 2. *Branch Points and Coincidences.*

If in a general (a, b) correspondence two or more points of a group of points corresponding to a given point P of a curve coincide, the point P is called a branch-point. The numbers of simple branch-points y and y' (that is those for which only two of the corresponding points coincide) in an (a, b) correspondence between curves of genera p and π are connected by the following relation, due to Zeuthen:*

$$y - y' = 2a(\pi - 1) - 2b(p - 1). \quad (1)$$

If the correspondence is involutorial, for example $(a, 1)$, then $y = 0$, and we have the theorem:

The number of double points of an involution of order a , genus π , on a curve of genus p is

$$y' = 2(p - 1) - 2a(\pi - 1). \quad (2)$$

If the correspondence between the curves is $(1, 1)$, then (1) furnishes a direct proof of Riemann's theorem concerning the equality of the genera of curves that are birationally equivalent. For, if $a = b = 1$, then $y = y' = 0$, and we have $2(\pi - 1) = 2(p - 1)$, hence $p = \pi$.

From Zeuthen's formula also follows† that there do not exist involutions of genus $p(p > 1)$ on curves of the same genus.

For, setting $\pi = p$ in (2),

$$y' = 2(p - 1) - 2a(p - 1).$$

But $y' \geq 0$, $\therefore p - 1 \geq a(p - 1)$, $\therefore a = 1$.

* H. G. Zeuthen, "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes." *Math. Annalen*, Vol. III (1870), pp. 150-156.

† Weber, *Jour. für Math.*, Vol. LXXVI (1876), p. 345.

§ 3. *Valence of Involutions.*

If the curves in (a, b) correspondence coincide, then we have an (a, b) correspondence between points of one curve. If as a point P (P') moves along the curve the b (a) corresponding points, together with the point P (P') counted γ times, ($\gamma \geq 0$), moves in a linear series, the correspondence is said to be of valence γ . In other words, given P and a group of b points corresponding to it. Let P go into P_1 , b into b_1 , then if the correspondence is of valence γ ,

$$\gamma P + b \equiv \gamma P_1 + b_1 \quad (\gamma \geq 0),$$

or the groups $\gamma P + b$, $\gamma P_1 + b_1$ belong to the same linear series of order $\gamma + b$. If γ is negative, (3) may not have a geometric meaning, but by transposing the term containing γ we get

$$\gamma P_1 + b \equiv \gamma P + b_1 \quad (\gamma > 0),$$

which can be interpreted to mean that $\gamma P_1 + b$, $\gamma P + b_1$ belong to the same linear series of order $\gamma + b$.

The number of coincidences z in a valence correspondence as given by the Cayley-Brill-Hurwitz formula is

$$z = a + b + 2p\gamma. \quad (3)$$

§ 4. *Notation.*

We shall represent involutions by small letters, with subscripts indicating the order and genus, and superscript 1 indicating the dimension, using letters of the Latin alphabet for rational involutions (linear series), and of the Greek alphabet for irrational involutions. Thus, $\gamma'_{a,\pi}$ reads: "An irrational involution of order a genus π ." g'_a —"a rational involution, or linear series, of order a ." Thus $g'_a \equiv \gamma'_{a,0}$. Capital letters are used to indicate individual groups: Γ_a, G_a .

§ 5. *Application to Cubic Curves.*

Every cubic has a single infinity of rational involutions of order 2. In fact, consider a pencil of lines with vertex at an arbitrary point S of the cubic. Each line of the pencil cuts the cubic in two points, each of which uniquely determines the other, since it fixes a line of the pencil. We have then an involution of pairs of points, and since the vertex S is arbitrary, we can construct a single infinity of such central involutions. The involutions are rational, for if we take the point S as the point $(0, 0, 1)$, for example, the equation of any line of the pencil is of the form $x + \lambda y = 0$. The coordinates of a point P on the cubic fix the value of the parameter λ , and we can express

the coordinates of the point P' , conjugate to P in the involution rationally in terms of the parameter.

In general, if the lines joining pairs of points of a simple involution of order 2 pass through a point or envelope a rational curve the involution is rational. We saw the truth of the first statement. In the second case the points of the curve enveloped by the lines are in (1, 1) correspondence with the groups of the involution, since to every point of the curve corresponds a line determining a group of the involution. The genus of the curve is, by definition, the genus of the involution. If the curve is rational, so is the involution.

We can construct on a cubic of genus 1 involutions of order 2 which are not rational by taking the product of two central involutions. Let us take a point $S(s)$, (we shall thus indicate the parameter, in terms of elliptic functions of which the coordinates of a point on the cubic can be expressed rationally) as center and project from it an arbitrary point $P(p)$ on the curve into $P_1(p_1)$. Project then $P_1(p_1)$ from another center $S_1(s_1)$ into $P_2(p_2)$. Repeat the process by projecting P from S_1 into $P'(p')$, and P' from S into $P''(p'')$. If $P_2=P''$, we have an involution of order 2 of which P and P_2 are a pair.

The necessary and sufficient condition that $P_2=P''$ is that the parameters of S and S_1 differ by half a period.

The sum of the parameters of three collinear points on an elliptic curve is congruent to zero.

The points S, P, P_1 are collinear, hence

$$s + p + p_1 \equiv 0, \text{ or } p_1 \equiv -(s + p).$$

The points P_1, S_1, P_2 are collinear, hence

$$-(s + p) + s_1 + p_2 \equiv 0, \text{ or } p_2 \equiv s + p - s_1.$$

Also the points P, S_1, P' lie on a straight line, hence

$$s_1 + p + p' \equiv 0, \text{ or } p' \equiv -(s_1 + p).$$

And the points P', S, P'' are collinear, hence

$$-(s_1 + p) + s + p'' \equiv 0, \text{ or } p'' \equiv s_1 + p - s.$$

If $P''=P_2$, $p''=p_2$, or $s + p - s_1 \equiv s_1 + p - s$,

$$\therefore 2s \equiv 2s_1 \pmod{\omega, \omega'},$$

hence the parameter of S_1 differs from that of S by half a period. Conversely, if s and s_1 differ by half a period, $P''=P_2$. Since the parameter of P does not enter in the last equation the statement is true for any point on the cubic.

If we draw the tangent to the cubic at S it will cut the curve again in one point $O(o)$. From O we can draw three tangents to the cubic, different from the tangent at O and OS . Let the points where the three tangents touch the curve be $S_1(s_1)$, $S_2(s_2)$ and $S_3(s_3)$. We have then the following four relations between the parameters of the points O, S, S_1, S_2, S_3 ,

$$2s + o \equiv 0, \quad 2s_1 + o \equiv 0, \quad 2s_2 + o \equiv 0, \quad 2s_3 + o \equiv 0.$$

Eliminating o we find that the parameters of S_1, S_2, S_3 differ from the parameter of S by half a period.

It can easily be seen that the converse is also true, namely, if the parameters of two points differ by half a period, the tangents to the cubic at these points intersect in a point on the curve.

Collecting the above results we can state that the necessary and sufficient condition that the product of two central involutions on a cubic of genus 1 is an involution of order 2 is that the tangents to the cubic at the centers of the involutions intersect in a point on the cubic.

Associated with any point $S(s)$ on the cubic there are three points $S_1\left(s + \frac{\omega}{2}\right)$, $S_2\left(s + \frac{\omega'}{2}\right)$, $S\left(s + \frac{\omega + \omega'}{2}\right)$ such that

$$SS_1 \equiv S_1S, \quad SS_2 \equiv S_2S, \quad SS_3 \equiv S_3S.$$

It is important to notice that we get the same three involutions, no matter where S is taken. In fact, let $SS_1 \equiv S_1S$, also $C(t)C_1(t_1) \equiv C_1(t_1)C(t)$. Under S a point $P(p)$ goes into $P_1(p_1)$, and P_1 under S_1 goes into $P_2(p_2)$, so that

$$s + p + p_1 \equiv 0, \quad s_1 + p_1 + p_2 \equiv 0.$$

Eliminating p_1 , we have

$$p + (s - s_1) + p_2 \equiv 0.$$

If the point P goes under C into $P'(p')$ and P' under C_1 goes into $P''(p'')$, we have:

$$t + p + p' \equiv 0, \quad t_1 + p' + p'' \equiv 0.$$

Hence, eliminating p' ,

$$p + (t - t_1) + p'' \equiv 0.$$

Since SS_1 , and CC_1 are, by hypothesis, involutions,

$$s - s_1 \equiv t - t_1.$$

Therefore $p_2 \equiv p''$, and the point P has the same conjugate in both involutions. The involutions are, therefore, identical.

The lines joining pairs of corresponding points envelope a curve of genus 1—the genus of elliptic functions in terms of which the equations of the lines can be expressed rationally. Hence, if the product of two central involutions on a cubic curve of genus 1 is an involution of order 2 its genus is 1.

§ 6. *General Theorems.*

Rational involutions have been studied in detail.* Comparatively little has been done in the field of irrational involutions. Castelnuovo† derived the following theorem: *A group of a $\gamma'_{a,\pi}$ on a curve of genus p has but $a-1$ conditions to belong to a group of a g'_n , if $n-r < p-a\pi$. For $\pi=0$ and $g'_n \equiv g_{2p-2}^{p-1}$ the theorem reduces to the Riemann-Roch theorem.*

Amodeo‡ derived from the Zeuthen and Cayley-Brill-Hurwitz formulas a number of theorems on the range of possible involutions on curves of given genus. In so far as the theorems refer to irrational involutions on curves of genus greater than 1, and of general moduli they are of no value, since, as will be pointed out later, such involutions do not exist.

In a later paper by Castelnuovo§ appears the important theorem:

The necessary and sufficient condition in order that a simply infinite series γ'_a of order a , and of index b , on a curve $F(x)=0$ of order n , genus p , belongs to a linear series g'_a of the same order, is that the series shall possess $2b(a+p-1)$ double points.

The proof, in brief, is as follows:

Construct on $F(x)=0$ a non-special linear series (that is a series g'_m , where $m-r=p$. It can be cut out by adjoints of order greater than $n-3$), of dimension $a-1$. The difference between the order and dimension of a non-special series being p , its order will be $a-1+p$. The series can always be selected in such a way that no given complete group of a points of γ'_a belongs to a group of the new series. For, we can choose $a-1$ points of a group of γ'_a , and add to them p points taken arbitrarily, but so as not to contain the a -th point of the group. We will have then a group of the series g'^{a-1}_{a-1+p} . But this one group will fix the series. Since every group will contain the same p points and $a-1$ other points, we will have constructed a series of the kind desired.

Applying the Segre formula|| we find that the number of groups of a points that the linear series g'^{a-1}_{a-1+p} and γ'_a have in common is

$$z=b(a+p-1)-1/2d,$$

* For the general theory on linear series see Clebsch-Lindemann, "Vorlesungen über Geometrie" (1876); Severi, "Lezioni di Geometria Algebrica" (1910, lith.). A list of references to the recent literature is given in Doehlemann's "Geometrische Transformationen," zweiter Teil (1908), p. 174.

† G. Castelnuovo, "Alcuni osservazioni sopra le serie irrazionali di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 4, Vol. VII² (1891), pp. 294-299.

‡ F. Amodeo, "Contribuzione alla teoria delle serie irrazionali involutorie giacenti sulle varietà algebriche ad una dimensione," *Ann. di Mat.*, s. 2, Vol. XX (1892), pp. 227-235.

§ G. Castelnuovo, "Sulle serie algebriche di gruppi di punti appartenenti ad una curva algebrica," *Rom. Acc. Lincei Rend.*, s. 5, Vol. XV (1906), pp. 337-344.

|| C. Segre, "Sulle varietà algebriche di una serie semplicemente infinita di spazi," *Rom. Acc. Lincei Rend.*, (4), Vol. III² (1887), pp. 149-153.

where d is the number of double points of γ'_a . It follows that

$$d = 2b(a + p - 1) - 2z.$$

Since $z \geq 0$,

$$d \leq 2b(a + p - 1).$$

If d has the value given by the equality sign, z will be equal to zero, and the linear series g_{a-1+p}^{a-1} will not contain any groups of γ'_a . But if a series g_{a-1+p}^{a-1} be constructed to contain in one of its groups a group of γ'_a (which is possible in ∞^{p-1} ways), then the series will contain γ'_a entirely.

Now construct another linear series, g_{a+p}^a , a group of which is to be made up of a group of γ'_a , Γ_a , and p arbitrary points: $c_1, \dots, c_p$. Then the series g_{a-1+p}^{a-1} , residual with respect to c_i will contain Γ_a , hence all other groups of γ'_a . In other words the groups residual to the groups of γ'_a with respect to g_{a+p}^a will all pass through c_i , where $i=1, 2, \dots, p$. Hence we have a group G_p residual to any Γ with respect to g_{a+p}^a . In consequence, γ'_a belongs to linear series g_a , which is residual to G_p with respect to g_{a+p}^a .

Conversely, if γ'_a belongs to a linear series g_a , then a linear series g_{a-1+p}^{a-1} which contain one group Γ will contain all. If a series g_{a-1+p}^{a-1} be constructed so as not to contain in any of its groups a given group of γ'_a , it will not contain any group of γ'_a . Hence z will be equal to zero, and $d = 2b(a + p - 1)$.

Stated in other words, the theorem given as the necessary and sufficient condition that an algebraic correspondence (a, b) between two curves of genera p, π can be expressed by means of a single equation (rational in the coordinates of corresponding points) is that the number of branch-points on one curve is $2b(a + p - 1)$, and on the other $2a(b + \pi - 1)$, and conversely.

If the correspondence is involutorial, for instance $(a, 1)$, then the series γ'_a is a rational series g'_a , if it has $2(a + p - 1)$ double points. This can be seen also from Zeuthen's formula, for, if in (2) we set $y' = 2(a + p - 1)$, we get $\pi = 0$; and, conversely, if $\pi = 0$, $y' = 2(a + p - 1)$. Thus, a central involution on a non-singular cubic, $(a=2, p=1)$ has four double points and is rational, while the involution obtained by taking the product of two central involutions has no double points and is of genus 1.

§ 7. *Statement of the Problem.*

The purpose of this paper is two-fold:

I. To find the range of all possible involutions on curves of given characteristics.

II. Given an involution, to determine the restrictions on a curve of given genus that it may possess this involution.

In order to ascertain the genus of an involution of order a with a given number of coincidences on a curve $F(x)=0$ of genus p , it will suffice to find any curve $f(y)=0$ in $(1, a)$ correspondence with $F(x)=0$. If the genus of $f(y)=0$ is π , the involution will be of genus π . The curve $F(x)=0$ can not have another involution of the same order and the same number of coincidences but different genus. For, suppose it has beside $\gamma'_{a,\pi}$ also $\gamma'_{a,\pi'}$, that is, let there be a curve $\phi(y')=0$ of genus π' also in $(1, a)$ correspondence with $F(x)=0$. Then,

$$2(p-1)-2a(\pi-1)=2(p-1)-2a(\pi'-1), \therefore \pi=\pi'.$$

We shall arrange involutions according to the genus of the curve $F(x)=0$. Every curve of genus p , $p=3\pi+(0, 1, 2)$, if not hyperelliptic, can be reduced to a curve of order not greater than $2\pi+2$, $2\pi+3$, $2\pi+4$ with $2\pi(\pi-1)$, $2\pi^2$, $2\pi^2+2\pi+1$ double points, and we shall, consequently, consider for every genus curves of the lowest order, the so-called normal curves. Hyperelliptic curves will be treated separately.

§ 8. *Involutions on Rational Curves.* $p=0$.

Setting $p=0$ in (2) we get:

$$y' = -2 - 2a(\pi - 1); \text{ since } y' \geq 0, -1 - a(\pi - 1) \geq 0, \\ \therefore a(1 - \pi) \geq 1. \text{ But } a > 0, \therefore \pi = 0.$$

Hence: *Irrational involutions do not exist on rational curves.**

An interesting application of this theorem is found by studying the asymptotic lines of certain ruled surfaces. Given a ruled surface of order $m+n$, having one m -fold directrix line and an n -fold directrix line. The genus of a plane section is

$$\frac{(m+n-1)(m+n-2)}{2} - \frac{m(m-1)}{2} - \frac{n(n-1)}{2} - \Psi = (m-1)(n-1) - \Psi,$$

where Ψ is the number of double generators.

The asymptotic lines are all algebraic and each belongs to a linear complex containing the congruence defined by the directrices. Every generator meets each asymptotic line in two points, and a plane section in one point. Hence, by our theorem we have examples of curves belonging to a linear complex that are not rational.†

*For a different proof of the same theorem see Lüroth, "Beweis eines Satzes über rationale Curven," *Math. Annalen*, Vol. IX (1876), p. 163.

† See C. P. Steinmetz, "On the Curves Which are Self-Reciprocal in a Linear Null System, and Their Configurations in Space," *AMERICAN JOURNAL OF MATHEMATICS*, Vol. XIV (1892), pp. 161-186; V. Snyder, "Asymptotic Lines on Ruled Surfaces Having Two Rectilinear Directrices," *Bulletin American Mathematical Society*, Vol. V (1899), pp. 343-353, and "Twisted Curves Whose Tangents Belong to a Linear Complex," *AMERICAN JOURNAL OF MATHEMATICS*, Vol. XXIX (1907), pp. 279-288. Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces" (1906), pp. 204-220.

§ 9. *Hyperelliptic Curves.*

Curves of genera 1 and 2 belong to the class of hyperelliptic curves, and it will be appropriate to take up at this point the study of involutions on hyperelliptic curves generally. This was done by Torelli,* and we shall reproduce his main results.

Irrational involutions on hyperelliptic curves are hyperelliptic. That is, if a curve $f(y)=0$ is in $(1, a)$ correspondence with a hyperelliptic curve $F(x)=0$ it is itself hyperelliptic. For, as is known, the $\gamma'_{a,\pi}$ on $F(x)=0$ is compounded with the g'_2 of the curve. The pairs of groups of $\gamma'_{a,\pi}$ conjugate under g'_2 form a rational involution on the double line of $F(x)=0$. The groups of the involution are in $(1, 2)$ correspondence with the points of $f(y)=0$. The latter, then, has a g'_2 , and is hyperelliptic.

In particular, curves of genera 1 or 2 can not have irrational involutions other than of genus 1. For, setting $p=1$ in (2), we have:

$$\begin{aligned} y' &= -2a\pi + 2a, \\ \pi &= 1 - \frac{y'}{2a}, \therefore \pi = 0 \text{ or } 1. \end{aligned}$$

If $p=2$,

$$\begin{aligned} y' &= 2 - 2a\pi + 2a, \\ \pi &= 1 - \frac{y'-2}{2a}, \therefore \pi = 0 \text{ or } 1. \end{aligned}$$

Given a hyperelliptic curve of order $2\pi+2$, genus π ,

$$y_1^2 = \prod_{i=1}^{2\pi+2} (x_1 - \alpha_i), \quad \alpha_i \neq \alpha_k. \quad (4)$$

Applying the transformation

$$x_1 = \frac{f(x)}{\phi(x)}, \quad y_1 = \frac{y}{[\phi(x)]^{\pi+1}}, \quad (5)$$

where $f(x)$, $\phi(x)$ are relatively prime polynomials of degree a we get the hyperelliptic curve

$$y = \prod_{i=1}^{2\pi+2} [f(x) - \alpha_i \phi(x)] = R(x), \quad (6)$$

where $R(x)$ is a polynomial of order $a(2\pi+2)$. The hyperelliptic curves (4) and (6) are in $(1, a)$ correspondence. The curve (6) has an involution (hyperelliptic) of order a , genus π , a $\gamma'_{a,\pi}$ which is represented on the x -axis ($y=0$) on which (6) is mapped doubly by the rational involution,

$$f(x) - x_1 \phi(x) = 0.$$

* R. Torelli, "Sulle involuzioni irrazionali nelle curve iperellittiche," *Palermo Rend.*, Vol. XIX (1905), pp. 297-304.

To determine the genus of the curve (6) we notice that the factors of $R(x)$ have no common roots. All or some of them may have multiple roots, of even or odd multiplicity. In that case

$$R(x) = [S(x)]T(x), \quad (6')$$

where $T(x)$ is a polynomial of order $2p+1$ or $2p+2$, with simple roots only. Transforming (6) birationally by means of

$$x=x', \quad y=y'S(x'),$$

we obtain

$$y'^2 = T(x) = \prod_{i=1}^{\substack{2p+1 \\ \text{or } 2p+2}} (x-b_i), \quad b_i \neq b_k. \quad (7)$$

(7) is of genus p . (6) is in $(1, 1)$ correspondence with it. Hence the genus of (6) is also p .

The coincidences of the curve (6) (which are the double points of g'_2) form $2\pi+2$ groups of a rational involution I_a on the x -axis which are made up of the roots of (7) each counted an odd number of times (≥ 1), and the roots of $S(x)$, different from the roots of $T(x)$ (6'), each counted an even number of times (≥ 2). We have at once the following theorem:

The necessary and sufficient condition in order that a hyperelliptic curve $F(x)=0$ of genus p contain an irrational (hyperelliptic) involution of order a , and genus π is that of the $2p+2$ coincidences on the line on which the curve is mapped doubly, each counted an odd number of times and, if necessary, with other points, each counted an even number of times it shall be possible to form $2\pi+2$ groups of a rational involution of order a .

The condition is sufficient. For, let the x -axis on which the hyperelliptic curve $F(x)=0$ is mapped doubly, contain an involution I'_a satisfying the given condition. Then if $f(x)-\alpha_1\phi(x)=0, f(x)-\alpha_2\phi(x)=0 \dots f(x)-\alpha_{2\pi+2}\phi(x)=0$ are the $2\pi+2$ groups, the curve $y^2 = \prod_{i=1}^{2\pi+2} [f(x)-\alpha_i\phi(x)]$ has the same group of coincidences as $F(x)=0$, and is rationally equivalent to it. Since the former curve is in $(a, 1)$ correspondence with $y^2 = \prod_{i=1}^{2\pi+2} (x-\alpha_i)$, $F(x)=0$ will also be in $(a, 1)$ correspondence with it, and will have a $\gamma'_{a,\pi}$.

Conversely, if $F(x)=0$ has a $\gamma_{a,\pi}$, it can be put in birational correspondence with $y^2 = \prod_{i=1}^{2\pi+2} [f(x)-\alpha_i\phi(x)]$ by a proper choice of the constants α_i , and the polynomials $f(x)$ and $\phi(x)$. The coincidences of the two curves on the x -axis will then be projective.*

*Segre, "Introduzione alla geometria sopra un ente algebrico semplicemente infinito," *Ann. di Mat.*, serie 2, Vol. XXII (1894), § 87, note.

But this is the condition stated in the theorem.

The particular case of cyclic irrational involutions on hyperelliptic curves has been studied by Wiman *

§ 10. *Non-Hyperelliptic Curves.*

Non-hyperelliptic curves of general moduli have only valence-correspondences,† hence they do not have irrational involutions. In order for a curve to possess an involution the moduli of the curve must be specialized. We shall commence the study of involutions on curves with specialized moduli with those of the lower order, namely, 2.

§ 11. *Irrational Involutions of Order 2.*

An irrational involution of order 2 on a given curve associates the points of the curve in pairs, so that to a point P corresponds a unique point P' , which is the conjugate of P in the involution. To the point P' corresponds the point P . Thus the involution defines a birational transformation of period 2 of the curve into itself. In other words, if a curve possesses a γ'_2 it must remain invariant under a birational transformation of period 2.

We shall consider first irrational involutions of period 2 on curves which remain invariant under the simplest of birational transformations—linear transformations.

§ 12. $p=3$.

The normal form of a non-hyperelliptic curve of genus 3 is a non-singular quartic. Since under any birational transformation that leaves the curve invariant the system of adjoints of order $n-3$, that is straight lines, goes over into a system of adjoints of the same order, the transformation is linear.

Consider a non-singular quartic invariant under the linear transformation:

$$L \equiv \begin{pmatrix} x_1 & x_2 & x_3 \\ -x_1 & x_2 & x_3 \end{pmatrix}^\dagger.$$

Its equation will be of the form

$$x_1^4 + x_1^2 \phi_2(x_2, x_3) + \phi_4(x_2, x_3) = 0, \quad (8)$$

where the ϕ_i 's are homogeneous polynomials in x_2 and x_3 of degree i .

* A. Wiman, "Über die hyperelliptischen Curven und diejenigen vom Geschlecht $p=3$ welche eindeutige Transformationen in sich zulassen," *Bihang till K. Svenska Vet. Akad. Handlingar*, Vol. XXI (1895).

† A. Hurwitz, "Über algebraische Correspondenzen und das verallgemeinerte Correspondenzprincip," *Math. Annalen*, Vol. XXVIII (1887), p. 560.

‡ Every linear transformation in the plane of period 2 can be put into this form. A. Hurwitz, "Ueber diejenigen algebraischen Gebilde, welche eindeutige Transformationen in sich zulassen," *Math. Annalen*, Vol. XXXII (1888), p. 290.

A straight line joining a pair of points P and P' conjugate in the involution will cut the quartic again in two points Q and Q' . Since the points P and P' interchange under the transformation which leaves the quartic invariant, the line PP' will go into itself. Hence, Q, Q' also interchange, in other words, Q and Q' are a pair of the involution. The point P determines not only its conjugate P' , but also another pair of the involution. Since P is arbitrary, every point on the quartic determines two pairs of the involution, or

If a curve of genus 3 has a γ'_2 , the involution is multiple.

Since we assume that C_4 is not hyperelliptic, it can not have a g'_2 . The lines joining pairs of points of the involution belong to a pencil with vertex at the center of homology $(1, 0, 0)$ not on the curve.

To find the genus of the involution we need only construct a curve in $(1, 2)$ correspondence with (8). By means of the transformation

$$T \equiv \begin{pmatrix} x_1 = \sqrt{y_1 y_3}, \\ x_2 = y_2, \\ x_3 = y_3, \end{pmatrix}$$

(8) goes over into the quartic

$$y_1^2 y_3^2 + y_1 y_3 \phi_2(y_2, y_3) + \phi_4(y_2, y_3) = 0, \quad (8')$$

which is in $(1, 2)$ correspondence with (8). (8') has a tacnode at the point $(1, 0, 0)$ and no other multiple points; it is, therefore, of genus 1.

An involution of order 2 on a non-hyperelliptic curve of genus 3 is of genus 1.

We can make use of Zeuthen's formula (2) to verify the result. $\gamma'_{2,1}$ on the quartic has four coincidences—the points of intersection of the line $x_1=0$ with the curve. If in (2) we put $y'=4, p=3, a=2$, we have

$$4=2(3-1)-2 \cdot 2(\pi-1), \text{ or } \pi=1.$$

§ 13. $p=4$.

Normal form of a non-hyperelliptic curve of genus 4 is a quintic with two double points. The equation of a quintic with two double points at $(0, 1, 0)$ and $(0, 0, 1)$ invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + x_1^2 \phi_3(x_2, x_3) + ax_2^3 x_3^2 + bx_2^2 x_3^3 = 0. \quad (9)$$

The center $(1, 0, 0)$ of the homology is a simple point on the quintic. A line joining a pair of points in the involution passes through $(1, 0, 0)$ and cuts the curve again in two points Q and Q' . By the same method as in the previous case we may therefore state the theorem:

If a curve of genus 4 has a γ'_2 , and is invariant under a linear transformation, the involution is multiple.

Under T (9) goes over into the quartic

$$y_1^2 y_3 \phi_1(y_2, y_3) + y_1 \phi_3(y_2, y_3) + a y_2^2 y_3 + b y_2^2 y_3^2 = 0. \quad (9')$$

(9') is of genus 2, since it has one double point at $(1, 0, 0)$.

An involution of order 2 on a curve of genus 4 is of genus 2.

The line $x_1=0$ cuts the quintic in one point besides the double points $(0, 1, 0)$ and $(0, 0, 1)$. The center $(1, 0, 0)$ is also a coincident point. Hence the involution has two coincidences. Setting in (2) $y'=2$, $p=4$, $a=2$, we have,

$$2=2(4-1)-2 \cdot 2(\pi-1), \text{ or } \pi=2.$$

§ 14. $p=5$.

Curves of genus 5, which do not possess a g'_3 can be reduced to a sextic with five double points. A curve of genus 5 having a g'_3 can be reduced to a quintic with one double point.

a. If a sextic remains invariant under L and does not pass through the center of homology $(1, 0, 0)$, its equation is of the form

$$x_1^6 + x_1^4 \phi_2(x_2, x_3) + \dots = 0.$$

If the center of homology is on the sextic it is a double point, for the equation of the curve is then of the form

$$x_1^4 \phi_2(x_2, x_3) + \dots = 0.$$

In either case, since lines passing through the center of homology and joining pairs of points conjugate in the involution, go over into themselves under the transformation which interchanges the points, every point on the sextic determines in the first case three, and in the second case two groups of the involution.

b. The equation of a quintic of genus 5 invariant under L is of the form

$$x_1^4 \phi_1(x_2, x_3) + \dots = 0,$$

the double point being on the axis of homology $x_1=0$. The center of homology $(1, 0, 0)$ is on the curve. A line joining a pair of points conjugate in the involution cuts the curve in another pair of points, which also belongs to the involution. We may conclude, then, that *if a curve of genus 5 has a γ'_2 , and is invariant under a linear transformation, the involution is multiple.*

§ 15. *Involution of Order 2 on Curves of Any Genus.*

In general, if a curve of any genus greater than 1, not hyperelliptic, and of any order, has a γ'_2 and is invariant under a linear transformation, the involution is multiple.

lution is multiple. For, suppose the order of the normal curve to which the given curve can be reduced by birational transformations is n . If n is even, $n=2m$, the normal curve either does not pass through the center of homology $(1, 0, 0)$, or has at the center a singular point of even multiplicity. The equation of the curve is of the form

$$x_1^{2(m-k)}\phi_{2k}(x_2, x_3) + \dots = 0,$$

where $k \geq 0$. Every line through the center of homology cuts the curve in $2(m-k)$ points which interchange in pairs under L . An arbitrary point on the curve thus determines $m-k$ groups of the involution.

If n is odd, $n=2m+1$, the equation of the normal curve is of the form

$$x_1^{2(m-r)}\phi_{2r+1}(x_2, x_3) + \dots = 0,$$

where $r \geq 0$. The center of homology is on the curve. Every line through it cuts the curve in $2(m-r)$ points. An arbitrary point on the curve determines $m-r$ groups of the involution.

§ 16. *Equations of Transformation.*

In order to determine the genera of involutions of order 2 on given curves, it is convenient to view the transformation which carries the given curve into a curve in $(1, 2)$ correspondence with it geometrically. If we consider a new plane (y_1, y_2, y_3) such that to any point (x_1, x_2, x_3) in A corresponds one point (y_1, y_2, y_3) in A' , but to any point (y_1, y_2, y_3) in A' correspond two points in A , we may write

$$T^{-1} \equiv \begin{pmatrix} x_1^2 = y_1 \\ x_2 x_3 = y_2 \\ x_3^2 = y_3 \end{pmatrix}, \text{ and } T \equiv \begin{pmatrix} x_1 = \sqrt{y_1 y_3} \\ x_2 = y_2 \\ x_3 = y_3 \end{pmatrix}.$$

T sets up a $(2, 1)$ correspondence between the plane A of the given curve and the plane A' of the curve in $(1, 2)$ correspondence with it. The fundamental elements in the A -plane are one fundamental point, $(0, 1, 0)$, and one fundamental line, $x_3=0$; in the A' -plane there are also one fundamental point, $(1, 0, 0)$, and one fundamental line, $y_3=0$. To the system of lines in the A -plane, $x_1+ax_2+bx_3=0$ corresponds in the A' -plane a system of conics $y_1 y_3 + (ay_2 + by_3)^2 = 0$ passing through the fundamental point $(1, 0, 0)$ and tangent to the fundamental line $y_3=0$. The basis points of the system of conics are the points of intersection of the conics $y_1 y_3 + ay_2^2$ and $y_3(y_1 + by_2) = 0$. The conics of the net have three-point contact at $(1, 0, 0)$, hence one free point of intersection which corresponds to a point of intersection of two lines in the A -plane.

To the system of lines in the A' -plane, $y_1 + ay_2 + by_3 = 0$, corresponds in the A -plane the system of conics $x_1^2 + ax_2x_3 + bx_3^2 = 0$. The conic $x_1^2 + ax_2x_3 = 0$ and the line-pair $x_3^2 = 0$ have two fixed points of intersection at the fundamental point $(0, 1, 0)$; the system of conics in the A -plane has therefore two free points of intersection, which correspond to a point of intersection of two lines in the A' -plane.

To a curve of order n (not passing through the fundamental point) generated in the A -plane by a point-pair P, P_1 , corresponds in the A' -plane a curve of order n counted twice. If the curve C_n in the A -plane passes k times through the fundamental point $(0, 1, 0)$, its image is a curve of order $n - \frac{k}{2}$ counted twice, together with the fundamental line $y_3 = 0$ counted k times.*

If C_n cuts the fundamental line $x_3 = 0$ in n distinct points, its image passes $\frac{n}{2}$ times through the fundamental point $(1, 0, 0)$. When two or more points of intersection coincide, that is if C_n has a multiple point on $x_3 = 0$, a corresponding number of tangents to the image curve at the point $(1, 0, 0)$ coincide.

If C_n does not pass through the center of homology $(1, 0, 0)$ its equation is of the form

$$x_1^{2m} + x_1^{2(m-1)}\phi_2(x_2, x_3) + \dots = 0,$$

$n = 2m$. If n is odd, $n = 2m + 1$, the center of homology is on C_n . Under T C_n goes over into

$$y_1^m y_3^m + y_1^{m-1} y_3^{m-1} \phi_2(y_2, y_3) + \dots = 0.$$

Hence, if a curve of order $2m$ in the A -plane does not pass through the point $(1, 0, 0)$, its image in the A' -plane has m branches touching each other at the point $(1, 0, 0)$.

The center of homology lies on C_n if n is odd, in which case C_n passes through it an odd number of times (≥ 1), or, if n is even, when the center of homology is a singular point on the curve of even multiplicity (≥ 2).

If C_n has a singular point of order k on the axis of homology $x_1 = 0$, we can without loss of generality take the point as the point $(0, 0, 1)$. If k is even, $k = 2s$, the equation of C_n is of the form

$$x_3^{n-2s} \phi_s(x_1^2, x_2^2) + \dots = 0.$$

By means of T we get as the image of C_n the curve

$$y_3^{n-2s} \phi_s(y_1, y_3, y_2^2) + \dots = 0,$$

* For a comprehensive treatment of birational transformations see K. Doehlemann, "Geometrische Transformationen," Vol. II (1908).

which has s branches touching the line $y_1=0$ at the point $(0, 0, 1)$. If k is odd, $k=2t+1$, the equation of C_n is of the form

$$x_2x_3^{n-(2t+1)}\phi_t(x_1^2, x_2^2) + \dots = 0,$$

and the image of C_n is the curve

$$y_2y_3^{n-(2t+1)}\phi_t(y_1, y_3, y_2^2) + \dots = 0,$$

which has t branches touching each other at the point $(0, 0, 1)$.

The nature of the transformation T requires that the singularities of the curve in the A -plane, barring those in one of the above-discussed exceptional positions, occur in even numbers of similar ones, one pair of which gives rise to one similar singularity on the image of the curve in the A' -plane.

§ 17. *Illustration.*

Let us now determine the genus of a γ'_2 on a sextic of genus 5 and not having a g'_3 , hence having five double points. Since the number of double points is odd, one of them has to be taken in an exceptional position, while the remaining four give rise to two double points on the image curve in the A' -plane. If we take the double point at the fundamental point, the image of the sextic will be a quintic. The sextic cuts the fundamental line in four points besides the fundamental point, hence two branches of the quintic touch each other at the fundamental point in the A' -plane. The latter singularity is equivalent to two double points. The quintic, then, has four double points, and is of genus 2. Algebraically we get the same result by noticing that the equation of a sextic not passing through the center of homology $(1, 0, 0)$ and having a double point at $(0, 1, 0)$ is of the form

$$x_2^4(ax_1^2+bx_3^2) + \dots + x_1^6=0.$$

Under T it goes over into the quintic

$$y_2^4(ay_1+by_3) + \dots + y_1^3y_3^2=0,$$

which has two consecutive double points at $(1, 0, 0)$.

Had we taken the double point on the axis of homology, the result would be the same. In fact consider a sextic with a double point at $(0, 0, 1)$. It cuts the fundamental line in six points. Its image is a sextic having three branches touching each other at the fundamental point in the A' -plane. This singularity is the equivalent of six double points. The sextic in the A' -plane has eight double points, it is of genus 2.

Consider now the case when the normal form of a curve of genus 5 is a quintic with one double point. The center of homology is on the quintic, and if we take the double point as the point $(0, 1, 0)$, the image of the quintic is a quartic with one double point, hence of genus 2.

An involution of order 2 on a curve of genus 5 is of genus 2.

§ 18. *General Involutions of Order 2.*

We can now proceed to the general case. Given C_n . In order to possess a γ'_2 its equation must remain invariant under some birational transformation of period 2. If the curve is non-singular, hence of maximum genus $= \frac{(n-1)(n-2)}{2}$, the transformation must be linear.* If n is even, the equation of the curve is of the form

$$x_1^n + x_1^{n-2}\phi_2(x_2, x_3) + \dots + \phi_n(x_2, x_3) = 0.$$

Its image under T is

$$y_1^{\frac{n}{2}} y_3^{\frac{n}{2}} + y_1^{\frac{n-2}{2}} y_3^{\frac{n-2}{2}} \phi_2(y_2, y_3) + \dots + \phi_n(y_2, y_3) = 0,$$

a curve of order n with two consecutive $\frac{n}{2}$ -fold points at the point $(1, 0, 0)$.

The genus of the curve is

$$\frac{(n-1)(n-2)}{2} - \frac{n}{2} \left(\frac{n}{2} - 1 \right) = \frac{(n-2)^2}{4}.$$

Let us apply as a check Zeuthen's formula. The number of coincidences of γ'_2 is n , the points of intersection of $x_1=0$ with the curve,

$$\therefore n = 2 \left[\frac{(n-1)(n-2)}{2} - 1 \right] - 4(\pi-1); \therefore \pi = \frac{(n-2)^2}{4}.$$

If we denote $\frac{n}{2}$, the number of groups of γ'_2 on a line through the center of homology by r , we can write the genus of the involution in the form $(r-1)^2$.

The genus of γ'_2 on a non-singular curve of even order is $(r-1)^2$, where r is the number of groups of γ'_2 on a line passing through one group of the involution.

If the order of the curve is odd, its equation is of the form

$$x_1^{n-1}\phi_1(x_2, x_3) + x_1^{n-3}\phi_3(x_2, x_3) + \dots + \phi_n(x_2, x_3) = 0.$$

Its image in the A' -plane is

$$y_1^{\frac{n-1}{2}} y_3^{\frac{n-1}{2}} \phi_1(y_2, y_3) + \dots + \phi_n(y_2, y_3) = 0,$$

a curve of order n . The genus of the curve is $\frac{(n-1)(n-3)}{4} = r(r-1)$,

where $r = \frac{n-1}{2}$, and is, as above, the number of groups of γ'_2 on a line containing one group.

* V. Snyder, "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1909).

The coincidences of γ'_2 are the n points of intersection of $x_1=0$ with the curve and the point $(1, 0, 0)$. By Zeuthen's formula,

$$n+1=2 \cdot \left[\frac{(n-1)(n-2)}{2} - 1 \right] - 4(\pi-1), \dots \pi = \frac{(n-1)(n-3)}{4}.$$

The genus of γ'_2 on a non-singular curve of odd order is $r(r-1)$, where r is the number of groups of γ'_2 on a line passing through one group of the involution.

Thus, on a non-singular quartic γ'_2 is of genus $(2-1)^2=1$, on a non-singular quintic of genus $2 \cdot (2-1)=2$, on a non-singular sextic of genus $(3-1)^2=4$, and so on.

If the given curve $F(x)=0$ is non-hyperelliptic, and has the equivalent of not more than $E\left(\frac{n-1}{2}\right)^2 - 3$ double points, $E(k)$ being the largest integer less than k , it remains invariant under linear transformations only.*

If the number of singularities is greater, the curve may remain invariant under transformations of period 2 other than linear. In that case, however, there exists a curve birationally equivalent to $F(x)=0$, the equation of which contains only even powers of one of the variables, say x_1 , $f(x_1^2, x_2, x_3) \equiv f^2(x)=0$, and which is therefore transformed into itself by L .†

Consequently, a curve possessing a γ'_2 is either itself invariant under L or can be put into $(1, 1)$ correspondence with a curve which is invariant under L . In the first instance, as we have seen, if $F(x)=0$ is not hyperelliptic, the involution is multiple. If the involution defines on $F(x)=0$ a transformation which is not linear, consider $f^2(x)=0$, identical with $F(x)=0$ as regards involutions. The pencil of lines through the center of homology cuts the curve in k pairs of points of the involution. If $k=1$, the involution is a rational g'_2 , and the curve is, therefore, hyperelliptic. If $k>1$, the involution is multiple. Hence we can generalize the theorem stated in § 15:

A curve having a simple γ'_2 is hyperelliptic. If a non-hyperelliptic curve has a γ'_2 the involution is multiple.

We can readily determine the genus of a γ'_2 on a curve of given characteristics. Consider $F(x)=0$, of genus p . The singularities of the curve, not in exceptional positions, appear in pairs of similar ones. They are equivalent to $2 \sum_{\substack{i=2, \dots \\ v=1, \dots}} v_i \cdot \frac{i(i-1)}{2}$ double points, where $2v_i$ denotes the number of i -fold points.

* V. Snyder, "On Birational . . .," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1909).

† A. Hurwitz, "Ueber diejenige . . .," Math. Annalen, Vol. XXXII.

They give rise on $f(y)=0$ in the A' -plane to singularities equivalent to $\sum_{\substack{i=2,\dots \\ v=1,\dots}} v_i \cdot \frac{8i(i-1)}{2}$ double points. If $F(x)=0$ has no other singularities, and does not pass through the center of homology $(1, 0, 0)$, $f(y)=0$ will in addition have $\frac{n}{2}$ branches touching each other at $(1, 0, 0)$. Its genus, and therefore the genus of the involution, is

$$\pi = \frac{2(n-1)(n-2) - 2\sum v_i \cdot i(i-1) - n(n-2)}{4}.$$

The discussion in § 15 has taken cognizance of all possible positions of singularities on $F(x)=0$. When the singularities of $F(x)=0$ are known the determination of the genus of the involution becomes a matter of numerical calculations.

§ 19. *Cyclic Involutions of Any Order.*

The transition to involutions of any order follows directly, if the involution determines a birational transformation of the curve into itself. If a point P goes into P_1 , P_1 into P_2 , $\dots$ and $P_a=P$ by the transformation, the involution is called *cyclic*. The transformation is always periodic, and is either linear, or the given curve $F(x)=0$ can be put into $(1, 1)$ correspondence with a curve, the equation of which is of the form $f(x_1^a, x_2, x_3)=f^a(x)=0$, and is therefore invariant under the linear transformation:

$$U \equiv \begin{pmatrix} x_1 & x_2 & x_3 \\ \theta x_1 & x_2 & x_3 \end{pmatrix}, \quad \theta^a = 1.*$$

If $F(x)=0$ has a γ'_a so does $f^a(x)=0$. The groups of the involution on $f^a(x)=0$ are cut out by a pencil of lines having its vertex at the center of homology $(1, 0, 0)$. If the lines of the pencil cut out only one group, γ'_a is a rational g'_a ; if they cut the curve in more than one group the involution is multiple. Hence,

A cyclic involution of order a on a given curve is either rational or, if irrational, is multiple.

A multiple cyclic involution may also be rational.

The genus of γ'_a can be determined in a manner exactly analogous to the one employed in the determination of the genus of an involution of order 2. The singularities of $F(x)=0$, not in exceptional positions, will have to appear in groups of a similar ones giving rise to $1/a$ -th, their number of similar singularities on $f(y)=0$. If $F(x)=0$ does not pass through the point $(1, 0, 0)$, $f(y)=0$ has $\frac{n(a-1)}{2}$ branches touching each other at $(1, 0, 0)$, and so on.

* A. Hurwitz, "Ueber diejenigen . . .," *Math. Annalen*, Vol. XXXII.

§ 20. *Non-Cyclic Involutions.*

Given a cone $K_{\Psi, \pi} = 0$ of order Ψ , genus π and a surface $F_a = 0$. The curve C of intersection goes b times through the vertex of the cone and is of order m , genus p , where $m = a\Psi + b$ and p is determined from Sturm's formula *

$$p = a(a-1)\Psi + (a-1)(b-1) + a\pi.$$

The curve C possesses an involution of order a , genus π . Through every point P of C passes a generator of K , which meets C in $a-1$ points besides P . But each generator of K meets but one group of a points, hence K, C are in $(1, a)$ correspondence.

If C is projected on a plane section γ of K , when the center of projection is at the vertex of the cone, C is projected a -fold on γ . If the center of projection 0 is on C but not at the vertex of the cone, C will be projected into a plane curve C_1 of order $m-1$, having $a-1$ branches with a common tangent at 0_1 the point in which the generator of K through 0 pierces the plane of projection. Let P be any point on C . The generator through P and the point 0 determine a plane which passes through the vertex of K , hence cuts it in $\Psi-1$ generators besides the one through 0 . The plane meets the plane of projection in a line through 0_1 and contains $\Psi-1$ groups of the involution. We have therefore an illustration of a multiple non-cyclic involution on C_1 . The image $\gamma'_{a, \pi}$ may be assumed at will, hence curves having involutions of any order can be constructed, which have a given curve for image of the involution.

The number of coincidences is the number of tangents to C through the vertex of the cone K . By the Cayley-Brill formula this is seen to be $2(a-1+p)$. This, by Castelnuovo's formula already cited (§ 6), is the maximum number an involution $\gamma'_{a, \pi}$ can have.

If the center of projection is on K but not on C the conditions are unchanged except that the vertex of the pencil of lines in the plane of C_1 is now an a -fold point at which all a branches have a common tangent. As before, each line of the pencil contains $\Psi-1$ groups of the involution.

Finally, if the center of projection 0 is not on K , the vertex 0_1 of the plane pencil is not on C_1 , and each line of the pencil contains Ψ groups of the involution.

Next, suppose we have a ruled surface $R_{n, \pi} = 0$ of order n , and π the genus of a plane section Γ . If $C_m = 0$ is a complete intersection of R and $F_k = 0$, $m = kn$, its characteristics are connected with those of R by the following relation:

$$m(k+n-2) = r + 2d, \quad (10)$$

* R. Sturm, "Ueber das Geschlecht von Curven auf Kegeln," *Math. Annalen*, Vol. XIX (1882), pp. 487-488.

where d is the order of the double curve on R and is given by

$$d = \frac{(n-1)(n-2)}{2} - \pi,$$

and r , the rank of C is $r = 2m + 2p - 2$. Substituting the values of r and d in (10) we get

$$m(k-1) - p - \delta = \frac{k(k-1)n}{2} - k(\pi-1) - 1, \quad (11)$$

δ being the number of times R and F touch.

If the complete intersection of R and F is made up of C and s generators, so that $m = kn - s$, we have

$$(m-s)(k+n-2) = r - r'.^*$$

r' , the rank of the system of generators, is zero, hence

$$(kn-2s)(k+n-2) = 2(kn-s) + 2p-2 + k(n-1)(n-2) - 2k\pi - 2s(n-2),$$

or, as above,

$$m(k-1) - p - \delta = \frac{k(k-1)n}{2} - k(\pi-1) - 1. \quad (11)$$

If the residual of C_m is another curve $C_{m'}$, and if there exists an $F_k = 0$ which cuts R in C_m or $C_{m'}$ and s generators then, by means of relation (11) we can find the genus of C_m or $C_{m'}$ in terms of the characteristics of R , then the genus of the other curve by means of (10), which reduces to the form of (11). Formula (11), which is a generalization of Sturm's formula due to Segre,† is applicable to any curve on a ruled surface.

If now C is projected into the plane curve C_1 , the k points in which each generator of R meets C will be projected into a group of k collinear points, but the same line contains $m-k$ other points, not belonging to a group. The lines containing each a group of the involution envelope a curve of class n , birationally equivalent to the dual of Γ . Each of the remaining points on a line, not belonging to the group on that line, belongs to a group on another tangent to the envelope.

In case the surface $F=0$ is also a ruled surface, on C are two distinct involutions, and hence also on C_1 . Let $F \equiv R'_{n',\pi'} = 0$ and $R_{n,\pi} = 0$ have j generators in common, so that $m = nn' - j$, $\delta = 2j$, $k = n'$, $k' = n$. From Segre's formula we have

$$(k-1)(nn'-j) - p - 2j = \frac{k(k-1)n}{2} - k(\pi-1) - 1,$$

$$(n'-1)(nn'-j) - p - 2j = \frac{k'(k'-1)n'}{2} - k'(\pi'-1) - 1.$$

* Salmon, "Analytic Geometry of Three Dimensions," fifth edition, Vol. I (1912), § 346, p. 358.

† C. Segre, "Recherches générales sur les courbes et les surfaces réglées algébriques," *Math. Annalen*, Vol. XXXIV (1889), pp. 1-25.

The number of coincidences in the first involution is $2(n'-1+p)$ and in the second is $2(n-1+p)$. The maximum genus of C when $j=0$ can be obtained from Salmon's theory of postulation.

§ 21. *Restrictions on the Moduli of a Curve Having an Involution.*

The application of the methods used in the preceding pages to the second question we set out to answer is immediate. Let us first consider a simple case. Given an involution of order 2 and genus 1, to find the simplest curve that can possess it; in other words, find the simplest curve upon which a non-singular cubic can be mapped doubly. Consider the cubic

$$y_2^2 y_3 = \phi_3(y_1, y_3). \quad (12)$$

By means of T^{-1} we find as its image in the A -plane the sextic

$$x_1^2 x_3^4 = \phi_3(x_1^2, x_3^2). \quad (12')$$

The sextic has a four-fold point and its genus is therefore 2. We notice that (12') is invariant not only under L , but also under another homology which replaces x_3 by $-y_3$, or that the existence of one elliptic involution of order 2 on a curve of genus 2 necessitates the existence of another involution of the same order and genus. In the same manner most of the theorems established by Torelli* by transcendental methods, can be proved as simple corollaries of the preceding theorems.

In general, in order to find the genus of a curve which possesses a $\gamma'_{a,\pi}$, we start in the A' -plane with a normal form of a curve of genus π . Every multiple point of the curve, not in an exceptional position, gives rise to a similar multiple point on the image curve in the A -plane. The procedure laid down in § 16 is followed in the determination of correspondents of multiple points in exceptional positions. The genus of the image curve is then easily calculated.

* R. Torelli, "Sulle curve di genere due contenenti una involuzione ellittica," *Rend. Acc. Napoli*, s. 3, Vol. XVII (1911), pp. 412-419.